NATIONAL COMMITTEE FOR MATHEMATICAL CONTESTS

British Mathematical Olympiad

Friday 20th November 1987

Time allowed - 3½ hours

PLEASE READ THESE INSTRUCTIONS CAREFULLY :

Write on one side of the paper only. Use a fresh sheet or sheets of paper for each question. Arrange your answers in order. On the first sheet of your script write ONLY your full name, age (in years and months), home address and school; do not put any working on this sheet. On every sheet of working write your name and initials clearly in capital letters, and the number of the question.

There is no restriction on the number of questions which may be attempted, but remember

USE FRESH SHEETS FOR EACH QUESTION

1. Find all the real solutions x of the equation

$$\sqrt{x + 1972098 - 1986\sqrt{x + 986049}} + \sqrt{x + 1974085 - 1988\sqrt{x + 986049}} = 1$$

where $\sqrt{\ }$ indicates the non-negative square root.

2. Find all the real-valued functions f defined on the set D of natural numbers $x \geq 10$ and satisfying the functional equation

$$f(x + y) = f(x)\, f(y)$$

for all $x, y \in D$.

3. Find a pair of integers r, s such that $0 < s < 200$ and

$$\frac{45}{61} > \frac{r}{s} > \frac{59}{80}.$$

Also prove that there is exactly one such pair r, s.

4. The triangle ABC has orthocentre H. The feet of the perpendiculars from H to the internal and external bisectors of angle BAC (which is not a right angle) are P and Q. Prove that PQ passes through the middle point of BC.

5. Numbers $d(n, m)$ with m, n integers, $0 \leq m \leq n$, are defined by

$$d(n, 0) = d(n, n) = 1 \quad \text{all } n \geq 0$$

and

$$md(n, m) = md(n-1, m) + (2n - m)\, d(n-1, m-1)$$

for $0 < m < n$. Prove that all the $d(n, m)$ are integers.

6. Show that the least positive value of

$$\frac{x^2 + y^2}{y},$$

where x, y are real numbers such that

$$7x^2 + 3xy + 3y^2 = 1,$$

is $\frac{1}{2}$.

REMEMBER : A FRESH SHEET FOR EACH QUESTION
WITH NAME AND QUESTION NUMBER ON EVERY SHEET